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THE USE OF FUNDAMENTAL GREEN'S FUNCTIONS FOR THE SOLUTION OF PROBLEMS OF HEAT CONDUCTION IN ANISOTROPIC MEDIA†

Y. P. CHANG, C. S. KANG and DAVID J. CHEN
Department of Mechanical Engineering, State University of New York, Buffalo, N.Y. 14214, U.S.A.

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Abstract—This paper is composed of two parts. The first part presents a short study on the fundamental Green's functions associated with heat conduction in anisotropic and isotropic media. Then by means of these functions and Green's formula, differential equations are transformed into integral equations. In the second part, these integral equations are solved for three specific problems in steady and transient states. The results are then compared with those of exact solutions and are found in good agreement, except for very small values of time. Effects of discontinuities of surface conditions and of boundaries are discussed in great detail. Anisotropic effects and the facility of the method are found to depend mostly on the determinant of conductivity coefficients which characterizes the type of differential equations.

NOMENCLATURE

c^*	specific heat;
h^*	heat transfer coefficient;
h,	h^*L^*/k_0^*, dimensionless heat transfer coefficient;
k_{ij}^*,	conductivity coefficients;
k_0^*,	reference conductivity;
k_{ij},	k_{ij}^*/k_0^*, dimensionless conductivity tensor;
k^{ij},	dimensionless resistivity tensor;
L^*,	reference length;
N,	number of spatial coordinates;
n^*,	outward drawn normal to surface;
n,	n^*/L^*;
Q^*,	rate of heat generation per unit volume;
Q,	$Q^*L^{*2}/k_0^*T_0^*$;
q^*,	heat flux;
q,	$q^*L^*/k_0^*T_0^*$;
T^*,	temperature;
T_0^*,	reference temperature;
T,	T^*/T_0^*;
t^*,	time;
t,	$t^*k_0^*/\rho^*c^*L^{*2}$;
x^*, y^*, z^*,	rectangular coordinates;
$x = x^*/L^*$,	$y = y^*/L^*, z = z^*/L^*$;
ρ^*,	density;
$\| \|$,	determinant, or matrix.

Subscripts

s,	pertaining to surface;
o,	reference quantities.

Superscript

*,	pertaining to physical quantities.

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INTRODUCTION

ANISOTROPIC media can occur in nature, such as woods, crystals and sedimentary rocks, and can also be produced artificially, such as laminated and fiber-reinforced construction and electronic materials, cables, cylinders, and tubes. Because of the rapid increase of their industrial use in recent years, the understanding of heat conduction in this type of material is of great importance. However, experimentally, it is difficult to make accurate measurements, and analytically, it is difficult to solve the differential equations.

This paper is concerned with the use of fundamental Green's functions* for the solution of problems of heat conduction in anisotropic and isotropic media. This method is well known in classical potential theory [1–3] and in theories of integral and partial differential equations [4–6] for investigating the existence and uniqueness of the solution of differential equations of parabolic and elliptic types.

The practical use of the classical, fundamental Green's functions together with the Green's second formula for the numerical solution of Laplace equation was probably first reported by Jawson [7]. By an approximate numerical technique, good results were reported by Sym [8] for a number of two-dimensional problems. Most recently, Rizzo and Shippy [9] solved the problem of heat conduction in an infinite cylinder of an isotropic medium by first applying the Laplace transformation and using the above method to solve the auxiliary equation. The numerical results were then inverted to yield the numerical solution of the original heat equation. Calculated data, however, are very limited. Since a circular contour is uniformly smooth, it is difficult to examine the accuracy of the method of solution. Shaw [10] investigated the heat conduction in a circular sector of an isotropic medium by the direct use of the classical fundamental Green's function but did not mention how good is the method for small values of time.

In general, it is difficult to obtain analytical solutions of problems of heat conduction in anisotropic media. Reported results have been, therefore, restricted to orthotropic cases, such as Giedt and Hornbaker [11] Touryan [12] and Chao [13]. Padovan [14] considered the heat conduction in a thin cylindrical shell of anisotropic media through the solution of an approximately formulated differential equation. One way of dealing with the anisotropic case is to transform the differential equation into the canonical form by changing the spatial coordinates. However, after the transformation, the domain will be deformed and rotated and surface conditions will become, in general, more complicated than original ones.

Since the purpose of using fundamental Green's functions for the solution of heat conduction problems in the present study is different from that in theory of integral equations, a short study on their basic properties is given in the beginning of this paper. By means of these properties and the Green's second formula, the heat conduction problems are then formulated in integral equations. These integral equations are particularly suitable for inverse problems which will be defined later. To illustrate the method and to investigate singularity and anisotropic effects, three systems with and without corners and with and without continuous boundary conditions are investigated in detail: a square prism, a circular cylinder and a hollow eccentric cylinder.

FUNDAMENTAL EQUATIONS AND ASSUMPTIONS

Consider an anisotropic medium in domain Ω bounded by surface S which may consist of n segments each being sufficiently smooth (in the sense of Liapunov [2], pp. 1–6). Physical and thermal properties of the medium are assumed constant. The heat conduction in the medium can then be formulated in general orthogonal curvilinear coordinates as follows:

$$\rho^* c^* \frac{\partial T^*}{\partial t^*} = -\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \xi_1}(h_2 h_3 q_1^*) + \frac{\partial}{\partial \xi_2}(h_1 h_3 q_2^*) + \frac{\partial}{\partial \xi_3}(h_1 h_2 q_3^*)\right] + Q^* \tag{1}$$

in Ω for $t > 0$, where

$$q_i^* = -\sum_{j=1}^{3} \frac{k_{ij}^*}{h_j}\frac{\partial T^*}{\partial \xi_j} \tag{2}$$

are the components of heat flux, ξ_i the general

* These functions are also called fundamental solutions, principal solutions, source functions, singular functions, or Green's functions in an infinite domain in texts of mathematics.

curvilinear orthogonal coordinates, h_i the scale factors, and other notations are defined in the Nomenclature. The determinant of the conductivity coefficients, $|k_{ij}|$ is assumed positive and definite, so that (1) is of the parabolic type and becomes elliptic type in steady state. In rectangular coordinates, (1) becomes especially simple and for this reason we shall use this coordinate system in the following presentation, unless stated otherwise. Using the dimensionless quantities defined in the Nomenclature, (1) takes the form,

$$\frac{\partial T}{\partial t} = \sum_{i,j=1}^{N} k_{ij} \frac{\partial^2 T}{\partial x_i \partial x_j} + Q(x_p) \qquad (3)$$

in Ω for $t > 0$, where x_p denotes the coordinates of any point in Ω, and N the number of spatial dimensions. The boundary conditions on T may be written in the form

$$T = F(x_p) \qquad \text{in } \Omega, \text{for } t = 0 \qquad (4)$$

$$\frac{\partial T}{\partial n^+} + h(x_s, t)T = f(x_s, t) \quad \text{on } S, \text{for } t > 0 \qquad (5)$$

where

$$\frac{\partial}{\partial n^+} = \sum_{i,j=1}^{3} k_{ij} \cos(n, x_i) \frac{\partial}{\partial x_j} \qquad (6)$$

x_s represents points on S, h and f are defined and continuous functions of x_s and t, $\cos(n, x_i)$ are the direction cosines of the normal n to surface S, and the initial condition $F(x_p)$ is a uniformly continuous function, or satisfying Hölder condition (see [3], pp. 7–9). In (5), either k_{ij} or h may be zero so that surface conditions of Dirichlet and Neumann types are included. The functions h and f may also depend on T.

For steady state, (3) takes the form

$$\sum_{i,j=1}^{N} k_{ij} \frac{\partial^2 T}{\partial x_i \partial x_j} = Q(x_p). \qquad (7)$$

With the boundary condition remains in the same form as (5) except that h and f are now independent of t.

It is to be noted from (1) and (2) that the problem of heat conduction in anisotropic media is three-dimensional, since, even though T may depend only on two spatial coordinates, the heat flux may be in three directions.

In the above formulation, the boundary conditions are assumed known while the temperature distributions in Ω for $t > 0$ are sought. This class of problems may be called the direct problem. Inversely, if the temperature history in Ω, but not on S, is known, while the rate of heat generation, initial temperature, surface temperatures and surface heat fluxes are to be found, then the problem may be referred to as an inverse problem. It is known that an inverse problem is more difficult to solve than the direct one [15, 16].

FUNDAMENTAL GREEN'S FUNCTIONS

Let $G_N(x_i, t\,|\,x_i', t')$ and $G_N(x_i\,|\,x_i')$ denote the fundamental Green's functions associated, respectively, with (3) and (7) for heat conduction in anisotropic media; and $g_N(x_i, t\,|\,x_i', t')$ and $g_N(x_i\,|\,x_i')$ those associated with heat conduction in isotropic media. To unify the presentation of these functions and to have them exhibit the same characters, we define

$$g_N(x_i, t\,|\,x_i', t') = \frac{1}{k[4\pi(t-t')]^{N/2}} e^{-r^2/4(t-t')} \qquad (8)$$

where

$$r^2 = \frac{1}{k} \sum_{i=1}^{N} (x_i - x_i')^2 \qquad (9)$$

and

$$g_3(x_i\,|\,x_i') = \frac{1}{4\pi r}, \quad g_2(x_i\,|\,x_i') = -\frac{1}{2\pi} \ln r \qquad (10)$$

where

$$r^2 = \sum_{i=1}^{N} (x_i - x_i')^2. \qquad (11)$$

The r's defined in (9) and (11) may be called geodesic distances. It is easy to see that $g_N(x_i, t|x_i', t')$ becomes singular at $x_i = x_i'$ and $t = t'$ and $g_N(x_i|x_i')$ at $x_i = x_i'$. These points are called poles. It can be readily verified by direct substitution that $g_N(x_i, t|x_i', t')$ satisfies

$$k\sum_{i=1}^{N}\frac{\partial^2 g_N}{\partial x_i^2} - \frac{\partial g_N}{\partial t} = 0 \text{ for } x_i \neq x_i', t \neq t'$$

$$g_N(x_i, t|x_i', t' = t) = 0 \tag{12}$$

and $g_N(x_i|x_i')$ satisfies

$$\sum_{i=1}^{N}\frac{\partial^2 g_N}{\partial x_i^2} = 0 \quad \text{for } x_i \neq x_i'. \tag{13}$$

Similarly, $G_N(x_i, t|x_i', t')$ and $G_N(x_i|x_i')$ are to satisfy, respectively,

$$\sum_{i,j=1}^{N} k_{ij}\frac{\partial^2 G_N}{\partial x_i \partial x_j} - \frac{\partial G_N}{\partial t} = 0 \quad \text{for } x_i \neq x_i', t \neq t'$$

$$G_N(x_i, t|x_i', t' = t) = 0 \tag{14}$$

and

$$\sum_{i,j=1}^{N} k_{ij}\frac{\partial^2 G_N}{\partial x_i \partial x_j} = 0 \quad \text{for } x_i \neq x_i'. \tag{15}$$

These can be achieved, if we write, in view of (8) and (10)

$$G_N(x_i, t|x_i', t') = \frac{|k^{ij}|^{\frac{1}{2}}}{[4\pi(t - t')]^{N/2}} e^{-R^2/4(t-t')} \tag{16}$$

where $|k^{ij}|$ is the inverse matrix to the matrix $|k_{ij}|$ and the geodesic distance R is defined by

$$R^2 = \sum_{i,j=1}^{N} k^{ij}(x_i - x_i')(x_j - x_j') \tag{17}$$

and

$$G_3(x_i|x_i') = \frac{|k^{ij}|^{\frac{1}{2}}}{4\pi R}, \tag{18}$$

$$G_2(x_i|x_i') = -\frac{|k^{ij}|^{\frac{1}{2}}}{2\pi}\ln R. \tag{19}$$

We now examine some basic properties of the fundamental Green's functions which will be used for isolating poles. For $x_i \to x_i'$, it can be easily shown by taking the limit that

$$\lim_{x_i \to x_i'} \int_0^t g_N(x_i, t|x_i', t')\,dt' = g_N(x_i|x_i') \tag{20}$$

$$\lim_{x_i \to x_i'} \int_0^t G_N(x_i, t|x_i', t')\,dt' = G_N(x_i|x_i'). \tag{21}$$

Let ε be the radius of a small spherical surface for $N = 3$, or of a small circle for $N = 2$, enclosing the point x_i in domain Ω, and let S_ε be the small surface. It can be easily shown that, for either steady or unsteady state,

$$\lim_{\varepsilon \to 0} \int_{S_\varepsilon} G_N\,dS_\varepsilon = \lim_{\varepsilon \to 0} \int_{S_\varepsilon} g_N\,dS_\varepsilon = 0 \tag{22}$$

and that

$$\lim_{\varepsilon \to 0} \int_{S_\varepsilon} \frac{\partial}{\partial n^+} G_N(x_i|x_i')\,dS_\varepsilon =$$

$$= \lim_{\varepsilon \to 0} \int_{S_\varepsilon} \frac{\partial}{\partial n} g_N(x_i|x_i')\,dS_e = -1$$

$$\lim_{\varepsilon = 0} \int_0^t dt' \int_{S_\varepsilon} \frac{\partial}{\partial n^+} G_N(x_i, t|x_i', t')\,dS_\varepsilon$$

$$= \lim_{\varepsilon \to 0} \int_0^t dt' \int_{S_\varepsilon} \frac{\partial}{\partial n} g_N(x_i, t|x_i', t')\,dS_\varepsilon = -1. \tag{23}$$

If the pole arises as $x_s \to x_s'$, we may isolate it by drawing a small hemispherical surface for $N = 3$ or a semi-circle for $N = 2$ of radius ε and center x_s, and we can obtain

$$\lim_{\varepsilon \to 0} \int_{S_\varepsilon} \frac{\partial}{\partial n^+} G_N(x_s|x_s')\,dS_\varepsilon$$

$$= \lim_{\varepsilon \to 0} \int_{S_\varepsilon} \frac{\partial}{\partial n} g_N(x_s|x_s')\,dS_\varepsilon$$

$$= \lim_{\varepsilon \to 0} \int_0^t dt' \int_{S_\varepsilon} \frac{\partial}{\partial n^+} G_N(x_s, t | x_s', t') \, dS_\varepsilon$$

$$= \lim_{\varepsilon \to 0} \int_0^t dt' \int_{S_\varepsilon} \frac{\partial}{\partial n} g_N(x_s, t | x_s', t') \, dS_\varepsilon = -\tfrac{1}{2}. \quad (24)$$

INTEGRAL EQUATIONS

We now wish to transform the differential equations (3) and (7) into integral equations by means of the fundamental Green's functions and Green's second formula.

Consider first the steady case. If L denotes the linear differential operator of (7) and $\bar{L}$ is its adjoint of (15), both taken with respect to x_i'

$$L[T(x_i')] = -Q(x_i'), \quad \bar{L}[G_N(x_i|x_i')] = 0 \quad \text{for} \quad x_i \neq x_i' \quad (25)$$

then Green's second formula takes the form

$$\int_\Omega \{G_N L[T] - T\bar{L}[G_N]\} \, d\Omega = \int_S \left[G_N(x_i|x_s') \frac{\partial}{\partial n^+} T(x_s') - T(x_s') \frac{\partial}{\partial n^+} G_N(x_i|x_s') \right] dS(x_s'). \quad (26)$$

Substituting (25) into (26), isolating the pole at $x_i = x_i'$ by a small sphere for $N = 3$, or a small circle for $N = 2$, of radius ε and taking the limit $\varepsilon \to 0$, we obtain

$$T(x_i) = \int_\Omega Q(x_i') \, G_N(x_i|x_i') \, d\Omega(x_i') + \int_S \left[G_N(x_i|x_s') \frac{\partial}{\partial n^+} T(x_s') - T(x_s') \frac{\partial}{\partial n^+} G_N(x_i|x_s') \right] dS(x_s'). \quad (27)$$

For the unsteady state, we can obtain by the same way

$$T(x_i, t) = \int_0^t dt' \int_\Omega Q(x_i', t') \, G_N(x_i, t|x_i', t') \, d\Omega(x_i') + \int_\Omega F(x_i') \, G_N(x_i, t|x_i', 0) \, d\Omega(x_i') + \int_0^t dt' \int_S \left[G_N(x_i, t|x_s', t') \frac{\partial}{\partial n^+} T(x_s', t') - T(x_s', t') \frac{\partial}{\partial n^+} G_N(x_i, t|x_s', t') \right] dS(x_s'). \quad (28)$$

For isotropic media, $T(x_i)$ and $T(x_i, t)$ are given by equations in the same forms of (27) and (28) with G_N replaced by g_N and $\partial/\partial n^+$ by $\partial/\partial n$.

From (27) and (28), it is seen that, if the boundary values, $(\partial T/\partial n^+)_s$ and T_s, are known, then the temperature distribution can be obtained by simply numerical integration. To evaluate these boundary values, however, we cannot set $x_i = x_s$ in these equations, because of the presence of poles at $x_s = x_s'$. Evaluating the singularity at $x_s = x_s'$ and using (24), we obtain

$$\tfrac{1}{2}T(x_s) = \int_\Omega Q(x_i') \, G_N(x_s|x_i') \, d\Omega(x_i') + \int_S \left[G_N(x_s|x_s') \frac{\partial}{\partial n^+} T(x_s') - T(x_s') \frac{\partial}{\partial n^+} G_N(x_s|x_s') \right] dS(x_s') \quad (29)$$

$$\tfrac{1}{2}T(x_s, t) = \int_0^t dt' \int_\Omega Q(x_i', t') \, G_N(x_s, t|x_i', t') \, d\Omega(x_i') + \int_\Omega F(x_i') \, G_N(x_s, t|x_i', 0) \, d\Omega(x_i')$$

$$+ \int_0^t dt' \int_S \left[G_N(x_s, t | x_s', t') \frac{\partial}{\partial n^+} T(x_s', t') - T(x_s', t') \frac{\partial}{\partial n^+} G_N(x_s, t | x_s', t') \right] dS(x_s'). \quad (30)$$

The boundary-value equations (29) for steady state or (30) for unsteady state together with the prescribed surface conditions (5) are sufficient to determine the unknown boundary values, $(\partial T/\partial n^+)_s$ and $(T)_s$.

Note that the volume integrals in (29) and (30) are known functions, and that, if the problem is three-dimensional in space, the boundary value equations are two-dimensional. Thus, the numerical solution of the boundary value equations requires less effort than the numerical solution of the original differential equation.

We now consider an inverse problem in domain Ω bounded by a surface of n segments. If temperatures at any $(n + 2)$ points in Ω are known, and we wish to find the temperature and heat flux at each surface segment, the rate of heat generation and the initial temperature, which are known only as uniformly distributed, then we have $(n + 2)$ equations of (28) and n equations of (30). These $(2n + 2)$ equations are sufficient to determine the $(2n + 2)$ unknowns. Thus, the integral-equation method is also a powerful one for the solution of inverse problems in anisotropic as well as in isotropic media. For instance, the inverse problems that were investigated in [16] can be readily written down in integral equations in the forms of (28) and (30) with G_N replaced by g_N.

NUMERICAL SOLUTIONS OF SOME TWO-DIMENSIONAL PROBLEMS

As a stringent test of the method of solution, we apply (27)–(30) to three specific problems in regular domains: (I) a long square prism, (II) a long circular cylinder and (III) a hollow eccentric cylinder. We assume that the surface condition is independent of the length coordinate, so that the temperature will depend only on two spatial variables. To facilitate the examination of the accuracy of the numerical results, we consider only surface conditions of Dirichlet type for which analytical solutions of problem (II) can be easily found. When an exact solution cannot be readily obtained, we shall specialize the integral equations to the isotropic case and compare the numerical results with those of the exact solution. For simplicity, we assume that heat generation is absent and that the medium is homogeneous in rectangular coordinates. It is to be noted that an anisotropic medium which is homogeneous in one coordinate system becomes heterogeneous in other coordinate systems. Suppose that the boundary conditions are, for the steady state,

$$T = f \qquad \text{on } S \quad (31)$$

and, for the transient state,

$$\begin{aligned} T &= f \qquad \text{on } S \text{ for } t > 0 \\ T &= 0 \qquad \text{in } \Omega \text{ for } t = 0. \end{aligned} \quad (32)$$

In view of the properties of fundamental Green's functions as indicated earlier, we may assume that $(\partial T/\partial n^+)_s$ and $(T)_s$ change very slowly in comparison with $(G)_s$ and $(\partial G/\partial n^+)_s$, respectively. In the numerical solution of the boundary value equations, we divide the contour S into N elements. Let $i = 1, 2 \ldots N$ denote the primary nodal points and $j = 1, 2 \ldots N$ the secondary nodal points. Then the boundary value equations (29) for steady problems can be written in the form

$$T(x_{si}, y_{si}) = 2 \sum_{j=1}^{N} \left(\frac{\partial T}{\partial n^+} \right)_{sj} A_{ij} - 2 \sum_{j=1}^{N} T(x_{sj}, y_{sj}) B_{ij} \quad (33)$$

where

$$A_{ij} = \int_{\Delta sj} G(x_{si}, y_{si} | x_s', y_s') \, dS(x_s', y_s') \quad (34)$$

$$B_{ij} = \int_{\Delta sj} \frac{\partial}{\partial n^+} G(x_{si}, y_{si} | x_s' y_s') \, dS(x_s', y_s') \quad (35)$$

and G is given by (19). Using matrix notation, (33) can be written as

$$(|1| + 2|B|)|T| - 2|A||Tn^+| = 0 \qquad (36)$$

where $|T|$ is the column matrix of elements $T(x_{sj}, y_{sj})$; $|T_{n^+}|$ that of elements $\partial T(x_{sj}, y_{sj})/\partial n^+$; $|A|$ and $|B|$ are square matrices of elements A_{ij} and B_{ij} respectively; and $|1|$ is the identity matrix.

The boundary-value equation (30) can be similarly written in algebraic equations. To save computer memory, the integration with respect to time may be performed stepwise, i.e. $T(x_{si}, y_{si}, t_m)$ is evaluated from $T(x_{si}, y_{si}, t_{m-1})$ where $m = 1, 2, 3, \ldots$ denote the nodal points of t.

$$T(x_{si}, y_{si}, t_m) = 2\sum_{j=1}^{N}\left(\frac{\partial T}{\partial n^+}\right)_{sj} A'_{ij} - 2\sum_{j=1}^{N}(T)_{sj}B'_{ij} + 2F_i \qquad (37)$$

where

$$F_i = \int_{\Omega} T(x', y', t_{m-1})\, G(x_{si}, y_{si}, t_m | x', y', t_{m-1}) \times \mathrm{d}\Omega(x', y') \qquad (38)$$

$$A'_{ij} = \int_{t_{m-1}}^{t_m} \mathrm{d}t' \int_{\Delta sj} G(x_{si}, y_{si}, t_m | x'_s, y'_s, t') \times \mathrm{d}S(x'_s, y'_s) \qquad (39)$$

$$B'_{ij} = \int_{t_{m-1}}^{t_m} \mathrm{d}t' \int_{\Delta sj} \frac{\partial}{\partial n^+} G(x_{si}, y_{si}, t_m | x'_s, y'_s, t') \times \mathrm{d}S(x'_s, y'_s)$$

and G is given by (16) with $N = 2$. Note that $T(x, y, t_{m-1}) = 0$ for $m = 1$. The integration with respect to time for A'_{ij} and B'_{ij} can be performed so that

$$A'_{ij} = \frac{|k^{ij}|^{\frac{1}{2}}}{4\pi}\int_{\Delta sj} E_i\left[\frac{R_s^2}{4(t_m - t_{m-1})}\right]\mathrm{d}S(x'_s, y'_s) \qquad (40)$$

$$B'_{ij} = \frac{|k^{ij}|^{\frac{1}{2}}}{2\pi}\int_{\Delta sj}\frac{D_s}{R_s^2} \times \exp\left[-\frac{R_s^2}{4(t_m - t_{m-1})}\right]\mathrm{d}S(x'_s, y'_s) \qquad (41)$$

where $Ei(z)$ is the exponential integral and

$$R_s^2 = k^{11}(x_{si} - x'_s)^2 + 2k^{12}(x_{si} - x'_s) \times (y_{si} - y'_s) + k^{22}(y_{si} - y'_s)^2$$

$$D_s = \cos(n, x'_s)\{k_{11}[k^{11}(x_{si} - x'_s) + k^{12}(y_{si} - y'_s)] + k_{12}[k^{12}(x_{si} - x'_s) + k^{22}(y_{si} - y'_s)]\} + \cos(n, y'_s)\{k_{12} \times [k^{11}(x_{si} - x'_s) + k^{12}(y_{si} - y'_s)] + k_{22}[k^{12}(x_{si} - x'_s) + k^{22}(y_{si} - y'_s)]\}.$$

Using matrix notation, (37) can be written in the same form as (36)

$$(|1| + 2|B'|)|T| - 2|A'||T_{n^+}| = 2|F|. \qquad (42)$$

The matrix (36) for the steady problem, as well as the matrix (42) for the transient problem, are solved together with the surface conditions:

$$T(0, y) = T(1, y) = T(x, 0) = 0 \qquad T(x, 1) = \sin \pi x \qquad (43)$$

for the square contour, and

$$T(1, \theta) = \begin{cases}\sin\theta & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi\end{cases} \qquad (44)$$

or

$$T(1, \theta) = a + b\cos\theta\, 0 < \theta < 2\pi \qquad (45)$$

for the circular contour, where a and b are constants. For the hollow cylinder, uniform and constant temperatures are assumed at the inner and outer surfaces. For the transient state, the above surface conditions are assumed to hold for $t > 0$.

In Figs. 1–3 some calculated results of temperature distributions in steady and transient states are shown for the three systems indicated above with surface-condition (44) for the circular

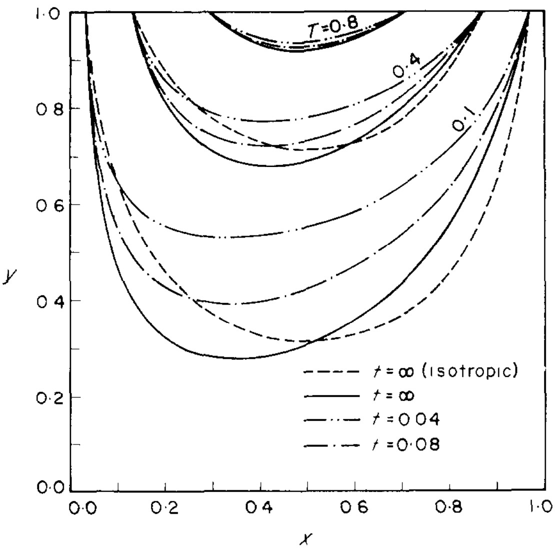

FIG. 1. Temperature distributions in a square, anisotropic medium. $k_{11} = 1$, $k_{12} = 0{\cdot}5$, $k_{22} = 1{\cdot}2$.

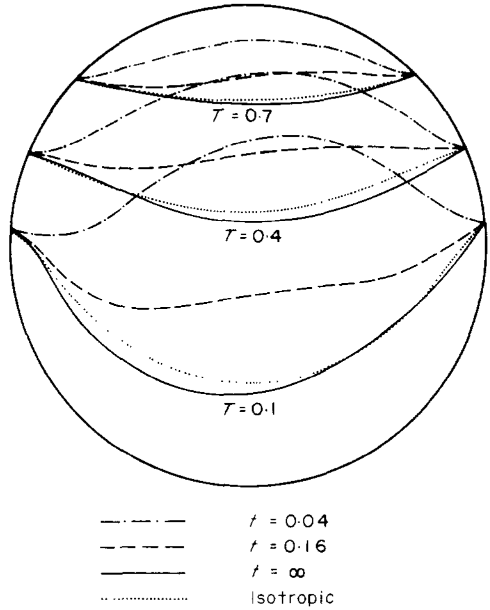

FIG. 2. Temperature distributions in a circular, anisotropic medium, $k_{11} = 1$, $k_{12} = 0{\cdot}5$, $k_{22} = 1{\cdot}2$.

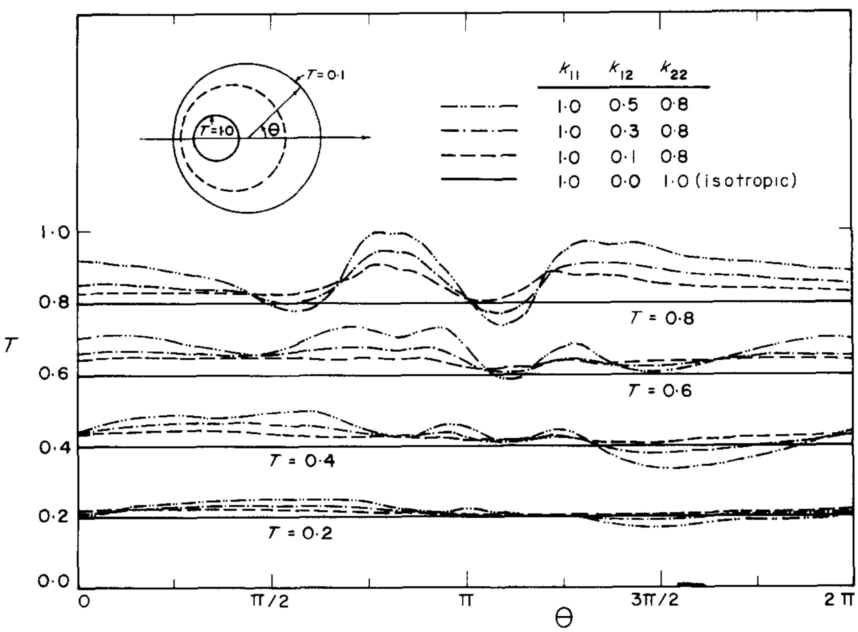

FIG. 3. Isotherms in anisotropic and isotropic media for various values of k_{12}.

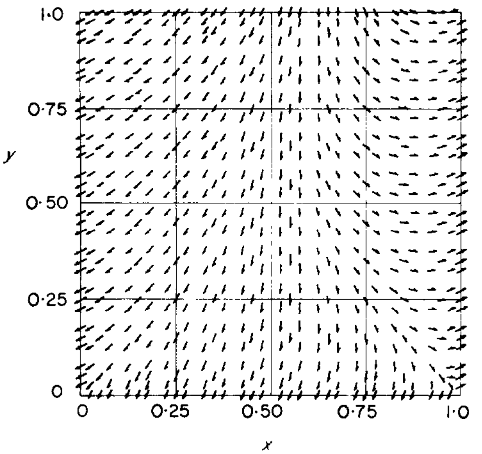

FIG. 4. Heat flux vectors in a square, anisotropic medium in steady state, $k_{11} = 1$, $k_{12} = 0{\cdot}5$, $k_{22} = 1{\cdot}2$.

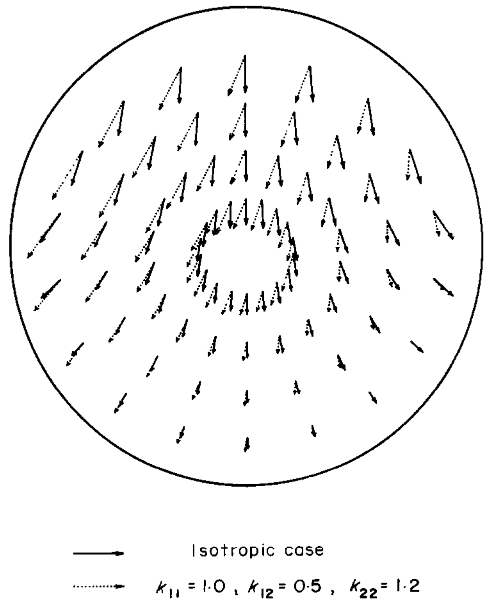

FIG. 5. Heat flux vectors in a circular, anisotropic medium in steady state.

cylinder. Distributions of steady heat flux vectors projected in xy-plane are shown in Figs. 4 and 5. Distributions of heat fluxes projected to the normal of contours are shown in Figs. 6 and 7. Some temperature distributions along a closed path parallel with the square contour are depicted in Fig. 8. The effects of anisotropy to temperature distributions and heat fluxes are clearly shown in these figures. The most significant quantity to characterize the anisotropy is the determinant of the conductivity coefficients, i.e. $|k_{ij}| = k_{11}k_{22} - k_{12}^2$. The smaller the value of $|k_{ij}|$, the more asymmetric are the temperature fields and heat flux

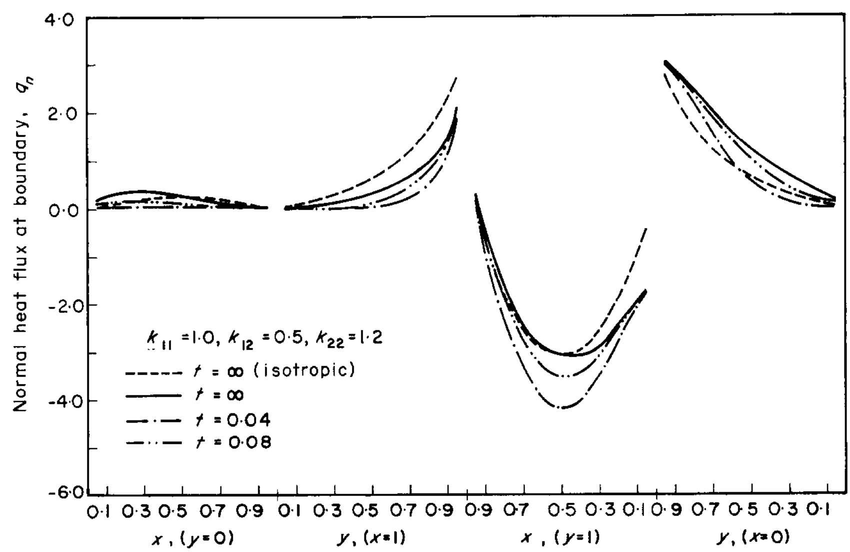

FIG. 6. Normal heat flux at square boundary.

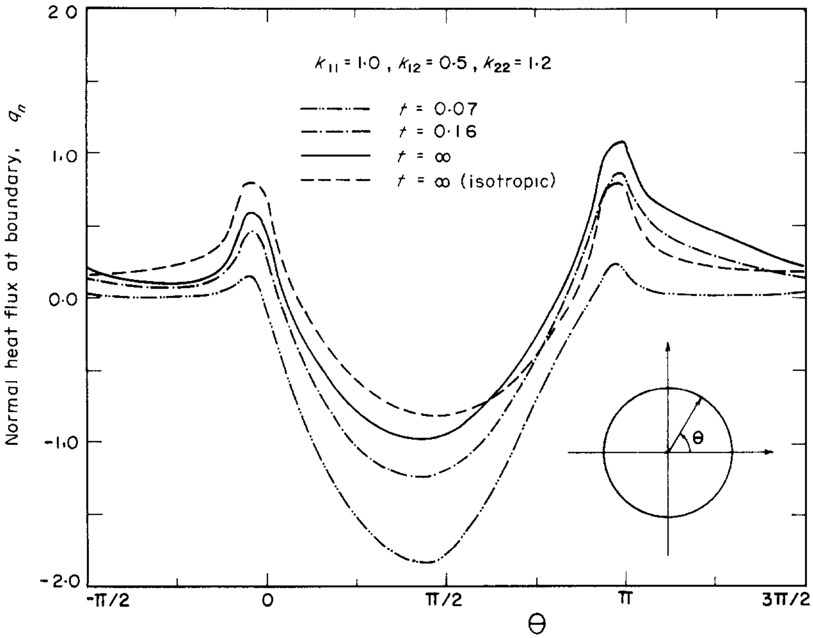

FIG. 7. Normal heat flux at circular boundary.

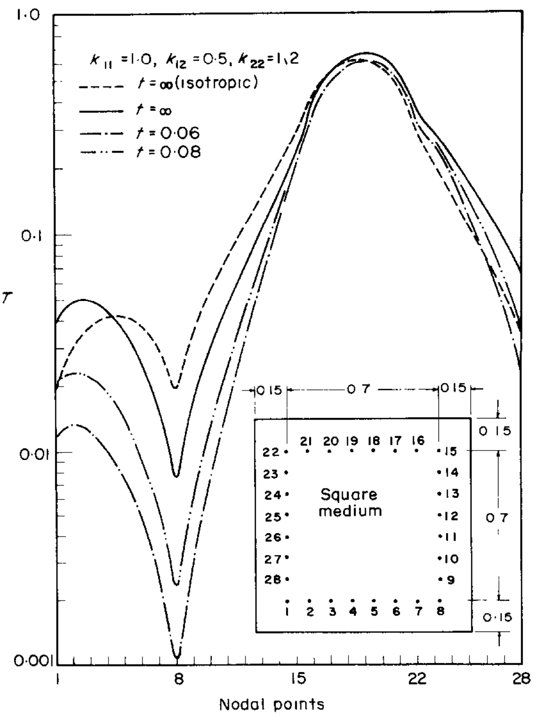

FIG. 8. Temperature distribution along a closed path.

vectors. Since the criterion $|k_{ij}| > 0$ determines the type of the differential equations, parabolic for transient problems and elliptic for the steady problems, therefore, the smaller the value of $|k_{ij}|$, the more difficult is the numerical calculation. From Fig. 4, it is clearly seen that heat flows out from the surface $y = 1$ near edge $(1, 1)$. For the circular domain, the outflow of heat occurs also near the point $(1, \pi)$ although it cannot be seen clearly from Fig. 5. Were the medium isotropic, inflow of heat would take place in these locations.

Calculated temperature distributions in steady state for the circular domain of anisotropic media with surface condition (45) is found identical with that of isotropic media for all values of k_{11}, k_{12}, k_{22}, a and b. This surprising result will be discussed later.

DISCUSSIONS

We now wish to examine the accuracy of the calculated results as shown in Figs. 1–8 for anisotropic media. Naturally, the simplest way is to compare the numerical results with those obtained by analytical means. Although it is not difficult to obtain analytical solutions of some of these problems, it is sufficient for the present purpose to consider only one case whose analytical solution can be readily written down.

For heat conduction in the circular cylinder, we consider the steady problem with surface condition (45). It is readily seen that the expression

$$T(r, \theta) = a + br\cos\theta \tag{46}$$

where (r, θ) are the polar coordinates, satisfies the boundary condition (45) and the differential equations expressed in rectangular coordinates for anisotropic as well as isotropic media. Since our numeral results from the integral equations have shown this identity, we can, therefore, conclude that the method of our numerical solution yields excellent results, for the circular cylinder with the continuous surface condition (45).

Table 1. Comparison between exact (upper figures) and numerical (lower figures) solutions for temperature distribution in a circular, isotropic medium with boundary condition (47)

.931	.938	.961		
.930	.937	.960		
.848	.855	.881	.928	
.844	.853	.879	.926	
.748	.754	.778	.829	.922
.743	.750	.776	.828	.923
.633	.635	.651	.689	.786
.626	.631	.648	.688	.787
.500	.512	.518	.531	.571
.500	.510	.516	.530	.567
.381	.374	.356	.315	.216
.374	.369	.352	.312	.213
.263	.254	.227	.173	.0785
.258	.250	.224	.172	.0774
.160	.150	.123	.0737	
.156	.147	.121	.0728	
.0722	.0641	.0402		
.0704	.0628	.0395		

To test results of other cases, we specialize (36) and (42) to those of isotropic media and results thus calculated are plotted in Figs. 1–8. In steady state, good agreement is obtained for the circular cylinder with surface condition (44) or (45). However, when the discontinuous surface condition

$$T(1, \theta) = \begin{cases} 1 & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi \end{cases} \tag{47}$$

is used, appreciable but tolerable errors are found near the points of discontinuity (i.e. $\theta = 0, \pi$), as can be seen from Table 1. For the square prism, a maximum error of 0·3 per cent is found near the corners as shown in Table 2. In transient state, for large times, a maximum error of about 0·3 per cent are also found around the points $(1, 0)$ and $(1, \pi)$ for the circular contour with surface condition (47) and around the four corners of the square contour. For $t = 0{\cdot}005$ error as large as 10 per cent in the temperature field are found for both the square and circular contours. The error decreases to about 1 per cent for $t = 0{\cdot}02$ and becomes negligible for $t > 0{\cdot}05$. Unfortunately, the large errors for small t cannot be reduced by taking smaller Δt, as was discussed in [15, 16]. An improved version of (42) was suggested in [16].

To understand the errors near the corners of the square contour and near the discontinuous point of the surface condition (47) on the circular contour, we have to go back to the theory of integral equations. When surface S is composed of a number of smooth segments, the corners are singular points. A discontinuity point of surface condition is, in effect, also a singular point. In obtaining the integral equations, we have tacitly excluded corners and assumed that the integral equation is valid for the truncated region. This is permissible in domain Ω only. For a corner, the term $\frac{1}{2}$ in (24) is to be changed to $1 - \omega/4\pi$ where ω is the solid angle of the corner, but this improves the accuracy only slightly. To reduce the error caused by edges of the contour, we may use "incomplete"

Green's functions, rather than the fundamental Green's function. When this method is used, the error for small t can be reduced to tolerable range. An incomplete Green's function is defined as the Green's function which satisfies a number of the boundary conditions but not all. To show this, we consider again the square prism and, for simplicity, we consider the isotropic case. Let $g_1(x, y|x', y')$ be the incomplete Green's function satisfying three homogeneous surface conditions at $x = 0$, $y = 0$ and $x = 1$, and $g_2(x, y|x', y')$ that satisfying only one homogeneous surface condition at $x = 0$. By the image method, g_1 and g_2 can be readily written down as follows:

$$g_1(x, y|x', y') = g(x, y|x', y') + \frac{1}{4\pi}\ln\frac{[(x+x')^2+(y-y')^2][(x-x')^2+(y+y')^2]}{(x+x')^2+(y+y')^2}$$

$$+\frac{1}{4\pi}\sum_{n=1}^{\infty}\ln\left\{\frac{[(x+x'-2n)^2+(y-y')^2][(x+x'+2n)^2+(y-y')^2]}{[(x+x'-2n)^2+(y+y')^2][(x+x'+2n)^2+(y+y')^2]}\right.$$

$$\left.\cdot\frac{[(x-x'-2n)^2+(y+y')^2][(x-x'+2n)^2+(y+y')^2]}{[(x-x'-2n)^2+(y-y')^2][(x-x'+2n)^2+(y-y')^2]}\right\} \tag{48}$$

$$g_2(x, y|x', y') = -\frac{1}{4\pi}\ln\frac{(x-x')^2+(y-y')^2}{(x+x')^2+(y-y')^2}. \tag{49}$$

Calculated results by using g_1 and g_2 are shown in Table 2. It is seen that results calculated by using g_1 are in good agreement with those of exact solution, except near the two corners (0, 1) and (1, 1) where the maximum error is about 0·3 per cent. Calculated results by using g_2 is better than those obtained by using $g(x, y|x', y')$ but worse than those obtained by using g_1.

The intolerable errors for very small t are evidently due to the approximate numerical technique. The smaller the time, the larger is the temperature slope near discontinuous points as shown in Fig. 8. Consequently, the approximation of $(T)_{s_t}$ and $(\partial T/\partial n^+)_{s_t}$ by constant values

Table 2. Comparison between exact and numerical solutions for temperature distribution in a square, isotropic medium: 1st row, exact; 2nd row, using g_1; 3rd row, using g_2; 4th row, using g

y \ x	0·05	0·15	0·25	0·35	0·45	0·55	0·65	0·75	0·85	0·95
Temperature gradients										
1·0	0·4933	1·432	2·230	2·810	3·115	3·115	2·810	2·230	1·432	0·4933
	0·4948	1·436	2·237	2·818	3·124	3·124	2·818	2·237	1·436	0·4948
	0·4948	1·436	2·237	2·818	3·124	3·124	2·818	2·235	1·441	0·4111
	0·4125	1·441	2·236	2·818	3·124	3·124	2·818	2·236	1·441	0·4125
Temperature distribution										
0·95	0·1336	0·3877	0·6039	0·7610	0·8435	0·8435	0·7610	0·6039	0·3877	0·1336
	0·1340	0·3879	0·6041	0·7612	0·8438	0·8438	0·7612	0·6041	0·3879	0·1340
	0·1340	0·3879	0·6041	0·7612	0·8438	0·8438	0·7612	0·6041	0·3878	0·1332
	0·1332	0·3878	0·6041	0·7613	0·8439	0·8439	0·7613	0·6041	0·3878	0·1332
0·85	0·0974	0·2826	0·4401	0·5546	0·6147	0·6147	0·5546	0·4401	0·2826	0·0974
	0·0973	0·2824	0·4400	0·5543	0·6144	0·6144	0·5543	0·4400	0·2824	0·0973
	0·0973	0·2824	0·4399	0·5543	0·6144	0·6144	0·5543	0·4399	0·2824	0·0968
	0·0968	0·2824	0·4399	0·5543	0·6145	0·6145	0·5543	0·4399	0·2824	0·0968

over small intervals of time and of length is no longer a valid simplification. To reduce the error caused by the approximate numerical technique, we may use the integral equation for the potential of a double layer [5]. For this purpose, we may consider again the isotropic case. Let $\mu(x_s, y_s, t)$ denote the potential density on the contour S which is defined for $t > 0$ and is zero for $t = 0$ and which satisfies the Hölder condition. Then we can seek the solution of $T(x, y, t)$ in the form

$$T(x, y, t) = \frac{1}{4\pi}\int_0^t \mathrm{d}t' \int_S \frac{\mu(x_s', y_s', t')}{(t - t')^2} \times \exp\left(-\frac{r^2}{4(t - t')}\right) \mathrm{d}S(x_s', y_s') \tag{50}$$

where μ is given by

$$\tfrac{1}{2}\mu(x_s, y_s, t) = f - \frac{1}{4\pi}\int_0^t \mathrm{d}t' \int_S \frac{\mu(x_s', y_s', t')}{(t - t')} \times \exp\left(-\frac{r^2}{4(t - t')}\right) \mathrm{d}S(x_s', y_s'). \tag{51}$$

Equation (51) can be solved by the method of iteration with the aid of the spline fit approximation. The main drawback of using potentials of a double or a single layer is that it will become more complicated in dealing with multiply connected, internally or externally bounded, regions, while (27)–(30) apply for any region. In view of the fact that the calculation for very small t is also difficult in the numerical solution of the original differential equation, we may, therefore, accept the present numerical method with the aid of the scheme suggested in [16] as a useful method for the solution of direct as well as inverse problems.

CONCLUDING REMARKS

The use of fundamental Green's functions together with the second Green's formula is a useful method for the solution of heat conduction problems in anisotropic media. In order to maintain reasonable accuracy, the following conditions are to be observed:

1. The bounding surface is, at least, piecewise smooth without acute corners.
2. The surface condition does not have abrupt change over each surface segment.
3. The determinant of conductivity coefficients is not too small.
4. Whenever the incomplete Green's function can be constructed, it should be used instead of the fundamental Green's function.

There are many advantages of using the fundamental Green's function G_N. The numerical solution of integral equations requires less effort than that of the differential equation. This is particularly true for irregular domains with complicated boundary conditions. Direct and inverse problems can be treated by the same way. Since $G_N(x_i, t|x_i', t')$ and $G_N(x_i|x_i')$ can be interpreted as instantaneous and steady heat sources in anisotropic media, solutions of a great many anisotropic problems, which would be very difficult to obtain by other methods, can be readily written down in the form of definite integrals which can be expressed in terms of tabulated functions for many cases of practical importance [17].

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L'UTILISATION DES FONCTIONS FONDAMENTALES DE GREEN POUR LA SOLUTION DE PROBLEMES DE CONDUCTION THERMIQUE DANS DES MILIEUX ANISOTROPES

Résumé—Cet article comprend deux parties. La première présente une courte étude des fonctions fondamentales de Green associée à la conduction thermique dans des milieux isotropes et anisotropes. Puis au moyen de ces fonctions et des formules de Green, les équations aux dérivées partielles sont transformées en équations intégrales. En seconde partie, ces équations intégrales sont résolues pour trois problèmes spécifiques en régimes permanents et transitoires. Les résultats comparés aux solutions exactes sont en bon accord sauf aux très petites valeurs du temps. Les effets de discontinuité des conditions de surface et des limites sont discutés de façon détaillée. On trouve que les effets de l'anisotropie et la facilité de la méthode dépendent surtout des coefficients de conductivité qui caractérisent le type des équations aux dérivées partielles.

GEBRAUCH DER FUNDAMENTALEN GREEN'SCHEN FUNKTION FÜR DIE LÖSUNG VON WÄRMELEITUNGSPROBLEMEN IN ANISOTROPEN MEDIEN

Zusammenfassung—Diese Arbeit besteht aus zwei Teilen. Der erste Teil enthält eine kurze Studie über die fundamentalen Green'schen Funktionen im Hinblick auf Wärmeleitung in anisotropen Medien. Dann werden mit Hilfe dieser Funktionen und der Green'schen Formel Differentialgleichungen in Integralgleichungen transfomiert.

Im zweiten Teil werden diese Integralgleichungen für drei spezifische Probleme stationärer und instationärer Art gelöst. Die Ergebnisse werden dann mit den exakten Lösungen verglichen. Sie stimmen außer für kleine Zeiten gut überein. Diskontinuitätseffekte der Oberflächenbedingungen und der Randbedingungen werden im Detail diskutiert. Die anisotropen Effekte und die Vorteile der Methode hängen ab von der Determinante der Wärmeleitungskoeffizienten, die den Typ der Differentialgleichung charakterisiert.

ИСПОЛЬЗОВАНИЕ ФУНДАМЕНТАЛЬНЫХ ФУНКЦИЙ ГРИНА ДЛЯ РЕШЕНИЯ ЗАДАЧ ТЕПЛОПРОВОДНОСТИ В АНИЗОТРОПНЫХ СРЕДАХ

Аннотация—Статья состоит из двух частей. Первая представляет собой краткое исследование фундаментальных функций Грина в теории теплопроводности для анизотропных и изотропных сред. Затем с помощью этих функций и формул Грина дифференцыльные уравнения преобразуются в интегральные. Во второй части эти интегральные уравнения решаются для трех частных задач в стационарном и нестационарном состониях. Сравнение полученных результатов с результатами точных решений показывает хорошее соответствие, за исключением очень небольших значений времени. Очень подробно обсуждается влияние разрывности условий на поверхности и границах. Найдено, что влияние анизотропности и простота метода зависят, в основном, от определителя коэффициентов теплопроводности, характеризующего тип дифференциальных уравнений.